Final exam for Kwantumfysica 1 - 2004-2005

Tuesday 19 April 2005 14:00 - 17:00

READ THIS FIRST:

- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Start each question (number 1, 2, 3) on a new answer sheet.
- The exam is open book. You are also allowed to use formula sheets etc.
- Note that this exam has 3 questions, it continues on the backside of the papers!
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or derive, you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- If you are ready with the exam, please fill in the **course-evaluation question sheet**. You can keep working on the exam until 17:00, and fill it in after shortly after 17:00 if you like.

Useful formulas and constants:

Electron mass	$m_e = 9.1 \cdot 10^{-31}$ kg
Electron charge	$e = -1.6 \cdot 10^{-19}$ C
Planck's constant	$h = 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar = 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between *x*-representation and *k*-representation of a state

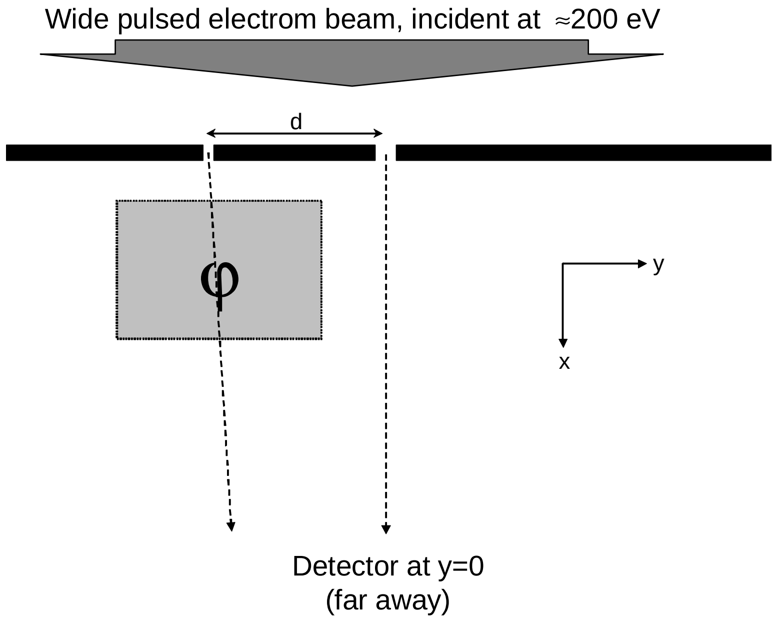
$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Z.O.Z.

Problem 1

Consider the following double slit experiment. A very wide electron beam (diameter of the Gaussian profile is much larger than *d*) is incident on a thin metal screen with two slits. The electrons come out of a pulsed source, which gives off electron ensembles with a Gaussian time-profile with a full-width-half-max of 100 ns at a repetition rate of 1 kHz. The electrons in the beam are accelerated and arrive with an average kinetic energy of 200 eV at the screen (source-screen distance is 3 m). The screen is at $x = 0$ m. The two slits have very sharp edges. The left slit is centered at $y = -100$ μm and has a width of $a_L = 0.1$ μm. The one on the right is centered at $y = 0$ μm and has a width of $a_R = 0.2$ μm. An electron detector is placed far from the screen at position $x = 2$ m, $y = 0$ m.



a) What is the average de Broglie wavelength of the electrons when they arrive at the screen? Make a rough estimate.

b) Estimate the uncertainty in the *x*-momentum of the electrons. Make a rough estimate.

c) Sketch the envelope of the normalized wavefunction (in *x*-representation, assume it can be represented by real values) that describes the *x*-position of the electrons while they are passing the screen. Explain your answer. You can use the rough estimates from the answers on **a)** and **b)**.

d) Sketch the normalized wavefunction (in *y*-representation, assume it can be represented by real values) that describes the *y*-position of the electrons in the beam for the moment that they just passed the screen. Explain your answer.

e) Calculate and sketch the normalized wavefunction (now in the k_y-representation) that describes the *y*-position of electrons that come out of the right slit, for the moment that they just passed the screen. You can use the sketches to make a rough estimates for the *y*-momentum uncertainty and *y*-position uncertainty for these electrons. Show that the *y*-position uncertainty and *y*-momentum uncertainty of these electrons is close to the minimum value that can occur.

f) You like to study quantum interference of electrons with this setup. You can change the phase φ of the electrons in the left beam that comes out of the screen with an apparatus in the gray area (see figure). You place the detector 2 meter after the screen at x = 2 m and y = 0 m. Is this far enough from the screen for observing interference? Show that this is the case, by evaluating whether the two partial electron beams have already a big overlap at the position of the detector. You can use rough estimates of the quantities that you need to answer this question.

g) Assume that the answer on **f)** is yes, and assume that the φ-controller does not reduce the intensity of the left beam. Sketch the observed intensity of the electron flux that hits the detector as a function of φ. Does the interference pattern have a maximum at $\varphi = 0$?

h) Consider the case where the intensity of the incoming electron beam towards the screen is reduced such that it has only one electron at the time in the volume between the electron-beam source and the detector. Can interference still be measured with this setup?

i) The φ-controller is formed by two plates that lie parallel to the x-y plane, closely above and under the left beam. The length in x-direction is 1 mm. By setting the same voltage V_φ on the plates you can control the potential energy eV_φ that the electrons experience between the plates. Estimate the value of V_φ that is needed for setting an extra phase difference between the beams of $\varphi = \pi$? Do you think this interference experiment is going to work in practice for these parameters? Explain your answer.

Problem 2
Consider a particle in one dimension in a time-independent potential $V(x)$. Write down the time-*dependent* Schrödinger equation for this system, and show how you can derive the time-*independent* Schrödinger equation for from this time-*dependent* Schrödinger equation.

Z.O.Z.

Problem 3

Consider a particle in a one-dimensional box with hard walls (infinitely high potential) at $x = -L/2$ and $x = L/2$. The Hamiltonian has eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$ with eigenvalues E_1 and E_2 respectively. An observable A with corresponding operator Â has eigenfunctions

$$\Psi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2)$$

$$\Psi_2 = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2)$$

and the two eigenvalues $a_1 \neq a_2$ that obey they eigenvalue equations $\hat{A}\Psi_n = a_n\Psi_n$ for $n = 1,2$. Assume Â contains no explicit time dependence.
At $t = 0$ the system is in $\Psi(x,0) = \Psi_1$

a) Derive the eigenfunctions $\varphi_1(x)$ and $\varphi_2(x)$ for this one-dimensional box problem.

b) Make a sketch of the initial state $\psi(x,0)$.

c) What is the probability of obtaining each of the following at $t = 0$?

i) a_1 when measuring A.
ii) a_2 when measuring A.
iii) E_1 when measuring energy.
iv) E_2 when measuring energy.

d) Determine $\psi(x,t)$ for $t > 0$.

e) Now find $\langle \hat{A} \rangle$ as a function of time.
Hint: Use Dirac notation for compact notation. Express $|\varphi_1\rangle$ and $|\varphi_2\rangle$ in terms of the eigenfunctions of Â, and determine first al terms $\langle \varphi_n | \hat{A} | \varphi_m \rangle$ for all relevant pairs n, m.

f) Does Â commute with the Hamiltonian? Explain your answer.

g) Suppose that the particle is still in the initial state, $\Psi(x,0)$, when the box instantaneously is doubled in size (that is, walls at $-L$ and L). Calculate the probability that you will find the particle in the ground state of the new box.

Useful relations for this question:

$$\cos a \cos b = \tfrac{1}{2}\cos(a-b) + \tfrac{1}{2}\cos(a+b)$$

$$\sin a \cos b = \tfrac{1}{2}\sin(a+b) - \tfrac{1}{2}\sin(a-b)$$

h) Suppose the particle is indeed in the ground state of the newly defined one-dimensional box of **g)**. At $t = t_0$, the potential energy that forms the walls is suddenly turned off, and the particle is free. Without doing any calculation, sketch how you expect the wave function to evolve at later times. Make a sketch for $t = t_0$ and for $t > t_0$, showing all important changes.